

PARTICLE CONTENT IN TOPOLOGICAL FIELD THEORIES¹

Andrew Toon²
University of Oxford
Department of Theoretical Physics
1 Keble Road
Oxford, OX1 3NP
England

Abstract

By demanding that the path integral measure of topological field theories be metric independent, we can derive powerful constraints on the particle content of a topological field theory as well as on the dimensionality of space-time.

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1 Introduction

Anomalies impose powerful constraints on quantum field theories. The well known examples being the necessity of the top quark in the standard model to cancel the triangle anomaly, and the critical dimensions of string theory to cancel the conformal anomaly. Both these examples illustrating the power of anomalies on the particle content of a field theory.

Anomalies are also expected to arise in topological quantum field theories. Arise in the sense that we wish to preserve topological invariance of these theories in the quantum theory which is manifest in the classical theory. The usual argument that shows topological invariance of, in particular, the partition function goes as follows [1]. Consider any general coordinate invariant field theory on some n -dimensional manifold M_n with some classical action S . The full gauged fixed quantum action S_q then takes the form:

$$S_q[\Phi_r, g_{ij}] = S[\Phi_r] + \delta_Q V[\Phi_r, g_{ij}]. \quad (1)$$

Here, Φ_r ($r = 1, 2, \dots$) are the fields of the theory including matter, gauge, ghost and auxiliary fields etc. The second term on the right hand side of equation (1) is the ghost plus gauge fixing term which is associated with all the gauge invariance of $S[\Phi_r]$. Note that the metric g_{ij} on M_n only appears in this term, that is one introduces the metric g_{ij} only in the process of gauge fixing. δ_Q stands for the related nilpotent BRS-transformation and thus the entire gauge fixing plus ghost term is written as a BRST variation of some functional $V[\Phi_r, g_{ij}]$. Naively, the partition function is then given by:

$$Z(M_n, g_{ij}) = \int D[\Phi] \exp iS_q[\Phi_r, g_{ij}]. \quad (2)$$

Following Witten's argument, topological invariance is preserved in the quantum theory since [1]:

$$\begin{aligned} \frac{\delta}{\delta g^{ij}} Z(M_n, g_{ij}) &= \int D[\Phi] \exp(iS_q) \frac{\delta}{\delta g^{ij}} (\delta_Q V) \\ &= \int D[\Phi] \exp(iS_q) \delta_Q \left(\frac{\delta}{\delta g^{ij}} V \right) = 0, \end{aligned} \quad (3)$$

by BRST invariance. What is crucial in the above derivation is the assumption that the path integral measure $D[\Phi]$ is metric independent. Once this is

established, it is clear from the above argument that topological invariance is preserved in the quantum theory. That is the partition function of the theory will be metric independent. However, as we will show, there exists a large class of “topological field theories” which have a metric dependent partition function but whose vacuum expectation value of some metric independent BRST invariant operator is topologically invariant.

The outline of this paper is as follows. In section 2, we will study the path integral measure in topological field theories and derive a constraint that preserves topological invariance in the quantum theory. We then use this constraint to investigate the particle content of topological field theories which have topologically invariant partition functions. We then discuss a large class of “topological field theories” which have metric dependent partition functions but whose vacuum expectation value of some metric independent BRST invariant operator is automatically metric independent.

2 Path integral measure in topological field theories

We will now show that in general the path integral measure of topological field theories is not metric independent. We will use the method of reference [2] who in turn exploit the method of Fujikawa [3].

Consider any general coordinate invariant field theory in n -dimensions with quantum action S_q given by equation (1). Recall that when one gauge fixes this theory, one picks a metric g_{ij} on M_n . The partition function is then not the naive partition function of equation (2) but is given by:

$$\tilde{Z}(M_n, g_{ij}) = \int \tilde{D}[\Phi] \exp iS_q[\Phi_r, g_{ij}], \quad (4)$$

where $\tilde{D}[\Phi]$ stands for an appropriate general coordinate invariant measure over the full set of fields Φ_r . Fujikawa [3] as proposed the following choice of the measure for being general coordinate invariant:

$$\tilde{D}[\Phi] = \prod_x \left[\prod_r d\tilde{\Phi}_r(x) \right], \quad (5)$$

where for any given field component Φ_r :

$$\tilde{\Phi}_r(x) = g^{\alpha r}(x) \Phi_r(x), \quad g(x) = \det g_{ij}. \quad (6)$$

The α_r are constants which depend upon the tensor nature of the fields as well as the space-time dimension. Some examples being:

$$\alpha_r = \begin{cases} \frac{1}{4} & \text{for each scalar field} \\ \frac{n-2}{4n} & \text{for each component of a covariant vector field} \\ \frac{n-4}{4n} & \text{for each component of a tensor field of rank two.} \end{cases} \quad (7)$$

It is now clear that:

$$\prod_x d\tilde{\Phi}_r(x) = \prod_x d[g^{\alpha_r}(x)\Phi_r(x)] = \prod_x [g^{\alpha_r\sigma_r}(x)d\Phi_r(x)], \quad (8)$$

where the signature σ_r is $+1$ (-1) for commuting (anti-commuting) fields. Thus, the general covariant Fujikawa measure becomes:

$$\tilde{D}[\Phi] = \prod_x [g(x)]^K \left(\prod_{r,y} d\Phi_r(y) \right), \quad (9)$$

where:

$$K = \sum_r \sigma_r \alpha_r, \quad (10)$$

is an index which measures the metric dependence of the path integral measure. The partition function (4) thus becomes:

$$\tilde{Z}(M_n) = \left(\prod_x [g(x)]^K \right) Z(M_n), \quad (11)$$

where $Z(M_n)$ is the partition function using the naive measure $\prod_{x,r} [d\Phi_r(x)]$. Thus for topological invariance to be preserved at the quantum level we require:

$$K = \sum_r \sigma_r \alpha_r = 0. \quad (12)$$

3 Particle content in topological field theories

We saw in the previous section that in order to preserve topological invariance of a classical topological theory in the quantum theory requires that $K = 0$. Note also that for supersymmetric type theories that $K \equiv 0$ and thus equation (12) gives no useful information about these types of theories.

Excluding such theories we will now use equation (12) to derive the particle content of topological field theories in various dimensions.

The simplest theory that we would like to write down is a gauge theory, with some gauge group G , together with a Lagrange multiplier which enforces some gauge constraint, ghost and anti-ghost fields. In n -dimensions, we then require:

$$K = \sum_r \sigma_r \alpha_r = \dim(G) \left(n \cdot \frac{(n-2)}{4n} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} \right) = 0. \quad (13)$$

Solving singles out $n = 3$ which corresponds to the Chern-Simons theory in 3 dimensions which has classical action:

$$S = \frac{1}{2} \int_{M_3} \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \rangle, \quad (14)$$

where A is a Lie algebra valued one form and $\langle \rangle$ denotes the trace over the gauge indices. It is interesting to note that the simplest topological gauge theory singles out the most complicated space topologically.

In view of the relation of the Chern-Simons theory to topological gravity let us for the moment stay in 3 dimensions and ask if one can couple topological matter to this theory with out ruining the topological invariance in the quantum theory. Consider then a scalar field which will contribute $1/4$ to K . To cancel this we need to couple it to a quantity that contributes $-1/4$ to K . This is achieved by coupling it to a rank two tensor field which contributes $3 \times (3-4)/4.3 = -1/4$. This is precisely the topological matter coupling considered in [4] with topological matter action given by:

$$S = \int_{M_3} \epsilon^{ijk} (\partial_i \phi) B_{jk}, \quad (15)$$

with ϕ being the scalar field and B_{ij} being a rank two anti-symmetric tensor field.

Let us now return to n dimensions. Consider again a gauge theory, with some gauge group G , together with a Lagrange multiplier, ghost and anti-ghost but now coupled to γ scalar particles in the adjoint representation of the gauge group. Equation (12) now gives:

$$K = \sum_r \sigma_r \alpha_r = \dim(G) \left(n \cdot \frac{(n-2)}{4n} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{\gamma}{4} \right) = 0. \quad (16)$$

Solving gives $n = 3 - \gamma$. The case $\gamma = 1$ giving precisely the particle content of 1+1 topological gravity with classical action given by [5]:

$$S = \frac{1}{2} \int_{M_2} \phi_A F_{\alpha\beta}^A \epsilon^{\alpha\beta}, \quad (17)$$

where ϕ_A is the scalar field in the adjoint representation and $F_{\alpha\beta}^A$ is the field strength of the gauge field. Again, staying with 2-dimensions, it is interesting to ask if one can couple topological matter fields to this theory. Again, let's consider a scalar field Φ (different from ϕ) which contributes $1/4$ to K . To cancel this we can couple it to an anti-symmetric tensor field, B_{ij} , of rank two which contributes $1 \times (2-4)/4.2 = -1/4$ to K . This has classical action:

$$S = \int_{M_2} \epsilon^{ij} \Phi B_{ij}, \quad (18)$$

where it should be clear that Φ and B_{ij} transform the same way under the gauge group G being considered. Note also that in [5] matter coupling of a scalar field to a vector field is considered which would imply $K \neq 0$. However, what happens under this coupling is that the symmetry of the theory is increased resulting in the need for further gauge fixing. It is precisely this further gauge fixing that restores $K = 0$.

4 Pseudo-topological field theories

The previous section concerned topological field theories whose partition function turned out to be topological invariants. However, there exists a large class of topological field theories whose partition function is not topological invariant but whose vacuum expectation value of some gauge invariant operator is. Consider for example the following theories on some odd dimensional manifold M_{2n+1} with classical action given by [6]:

$$S = \int_{M_{2n+1}} \Omega_{2n+1}, \quad (19)$$

where Ω_{2n+1} is the generalised Chern-Simons $2n + 1$ -form:

$$\Omega_{2n+1} = (n + 1) \int_0^1 dt \langle A(tdA + t^2 A^2)^n \rangle, \quad (20)$$

with A being a Lie algebra valued one-form. Since, just like the Chern-Simons 3-form, the classical action (19) is independent of any metric on M_{2n+1} , one would expect to obtain only topological information from these field theories. However, to quantise this theory we note that we have a single gauge field and thus we need to introduce a Lagrange multiplier to gauge fix plus the corresponding ghosts and anti-ghosts. Using equation (12) we see that $K = 0$ only in 3-dimensions. Thus, it appears that the generalised Chern-Simons theory will have a “topological anomaly” for all odd dimensions different from 3. There also appears to be a corresponding problem with the even dimensional models considered in [6] where a single scalar field is coupled to the field strength of some gauge field. We see this from equation (12) which seems to imply a “topological anomaly” in all even dimensions different from 2.

The above models may not in general have topological invariant partition functions but their vacuum expectation value of some gauge invariant quantity will for the following reasons. We now assume that $K \neq 0$ and consider the vacuum expectation value of any classically topologically invariant quantity $W[\Phi_r]$. We thus have:

$$\langle \tilde{W} \rangle = \frac{\int \tilde{D}[\Phi] W \exp iS_q}{\int \tilde{D}[\Phi] \exp iS_q} = \frac{[\prod_x g(x)]^K \int D[\Phi] W \exp iS_q}{[\prod_x g(x)]^K \int D[\Phi] \exp iS_q} = \langle W \rangle, \quad (21)$$

where $\langle W \rangle$ is the vacuum expectation value of W using the naive measure $\prod_{x,r} [d\Phi_r(x)]$. Thus, for non zero K , we see that the metric dependence cancels giving a metric independent result which follows from the argument given by equation (3). Thus if $K \neq 0$ we see that one may still obtain topological information even though the partition function is not a topological invariant. See for example reference [7] where the 5-dimensional generalised Chern-Simons term is discussed and its relation to link invariants.

5 Conclusion

We have seen that by demanding the path integral measure in topological field theories be metric independent, we are led to powerful constraints on the particle content of a theory as well as on the dimensionality of space-time. These constraints being particularly powerful if we want the theories partition function to be a topological invariant.

As we have seen, given a classical action which is metric independent, we can have two types of topological field theories. One with $K = 0$ and the other with $K \neq 0$, which we called pseudo-topological field theories because their partition function turns out to be metric dependent but their vacuum expectation value of some gauge invariant operator is not. One is now led to the question wheather the $K \neq 0$ models might have possible applications to physics. This being a natural question since the main problem of taking topological field theories seriously as physical models is how can some metric structure arise in these theories?

The general idea of how topological field theories may find application to physics goes as follows. One imagines that the initial state of the universe is in an unbroken phase. That is the metric of space-time is zero and thus there being no notion of physics as we know it. It is this phase that is described by a topological field theory. Once this view is accepted, the most fundamental problem becomes why is the space-time metric we see today nondegenerate instead of zero? One would expect some kind of Higg's mechanism to come to our rescue here as was done in [8,9] but to do this one must impose some hidden structure on the unbroken phase of the theory. That is, for example [9], start of with matter fields already in the theory, show that the theory admits an unbroken phase (show that the unbroken phase is a solution to the classical equations of motion) and then argue that a possible vacuum instability may occur leading to nondegenerate metrics. The problem then with a Higg's like mechanism is that one must start with a theory that already has physical structure but admits a diffeomorphism invariant phase. This really running counter with the notion of topological field theories since the initial phase they are supposed to describe has no metric structure and thus no physics. It really describes a dead or unborn universe.

The pseudo-topological field theories, therefore, might have a natural solution to the above problem. That is, the unbroken phase of gravity being described by the vacuum expectation value of some operator which we know to be metric independent. The broken phase then being the simplest "observable" in the theory, namely the partition function which is metric dependent. This idea at least confirming many beliefs that physics should be simple.

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References

- [1] E. Witten, Commun. Math. Phys. 117 (1988) 353.
- [2] R. Kaul, R. Rajaraman, Phys. Lett. B249 (1990) 433.
- [3] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195
- [4] J. Gegenberg, G. Kunstatter, H. Leivo, Phys. Lett. B252 (1990) 381.
- [5] A. Chamseddine, D. Wyler, Nucl. Phys. B340 (1990) 595.
- [6] A. Chamseddine, Nucl. Phys. B346 (1990) 213.
- [7] P. Gusin, Mod. Phys. Lett. A, Vol. 7, No. 34 (1992) 3203.
- [8] R. Percacci, Nucl. Phys. B353 (1991) 271.
- [9] S. Giddings, Phys. Lett. B268 (1991) 17.